3.2 Multiple recurrence for weak mixing actions

Definition 3.2.1. Suppose $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is a measure preserving action of a countable amenable group Γ on a probability space $(X, \mathcal{B}, \mu)$. Let $F_n \subset \Gamma$ be a Følner sequence, and suppose $A \subset L^\infty(X, \mathcal{B}, \mu)$ is a Γ-invariant unital, self-adjoint subalgebra. A state φ on A is said to be **generic** with respect to F_n, and μ if

$$\lim_{n \to \infty} \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \varphi(\sigma_\gamma(f)) = \int f \, d\mu,$$

for all $f \in A$.

The example to keep in mind is when $\mathbb{Z} \curvearrowright (\mathbb{T}, \mathcal{B}_{\text{orel}}, \mu)$ is the rotation action of $\mathbb{Z}$ on the circle by an irrational (modulo 2π) angle. Where A is the algebra of continuous functions on $\mathbb{T}$, and $\varphi \in A^*$ is given by evaluation at some point $z_0 \in \mathbb{T}$. Then it follows from Birkoff's Ergodic Theorem that for almost all choices of z_0, this state is generic. This justifies the terminology.

Lemma 3.2.2. *Suppose $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is an ergodic, measure preserving action of a countable amenable group Γ on a probability space $(X, \mathcal{B}, \mu)$. Let $F_n \subset \Gamma$ be a Følner sequence, and suppose $A \subset L^\infty(X, \mathcal{B}, \mu)$ is a Γ-invariant unital, self-adjoint subalgebra. If φ is a state on A which is generic with respect to F_n, and μ then*

$$\lim_{n \to \infty} \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \sigma_\gamma(f) = \int f \, d\mu,$$

for all $f \in A$, where the convergence is in $L^2(A, \varphi)$.

Proof. Note, that when φ is given by $\varphi(f) = \int f \, d\mu$ then this is just a restatement of von Neumann's Ergodic Theorem.

Let $\varepsilon > 0$ be given, and suppose that $f \in A$. Note that by subtracting the integral we may assume that $\int f \, d\mu = 0$. Since the action is ergodic it follows from von Neumann's Ergodic Theorem that there exists $n_0 \in \mathbb{N}$ such that

$$\| \frac{1}{|F_{n_0}|} \Sigma_{\gamma_0 \in F_{n_0}^{-1}} \sigma_{\gamma_0}(f) \|^2_{L^2(X, \mathcal{B}, \mu)} < \varepsilon.$$

Using the triangle, followed by the Cauchy-Schwartz inequality $((|\frac{1}{N}\Sigma_{n=0}^{N-1} a_n|)^2 \leq (\frac{1}{N}\Sigma_{n=0}^{N-1}|a_n|)^2 \leq \frac{1}{N}\Sigma_{n=0}^{N-1}|a_n|^2)$, and then using the fact that φ is generic we then have

$$\limsup_{n \to \infty} \varphi(| \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \frac{1}{|F_{n_0}|} \Sigma_{\gamma_0 \in F_{n_0}^{-1}} \sigma_{\gamma\gamma_0}(f)|^2)$$

$$\leq \limsup_{n \to \infty} \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \varphi(\sigma_\gamma(| \frac{1}{|F_{n_0}|} \Sigma_{\gamma_0 \in F_{n_0}^{-1}} \sigma_{\gamma_0}(f)|^2))$$

$$= \| \frac{1}{|F_{n_0}|} \Sigma_{\gamma_0 \in F_{n_0}^{-1}} \sigma_{\gamma_0}(f) \|^2_{L^2(X, \mathcal{B}, \mu)} < \varepsilon.$$

Also, for each $\gamma_0 \in F_{n_0}$ we have that

$$\limsup_{n\to\infty} \| \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} (\sigma_\gamma(f) - \sigma_{\gamma\gamma_0}(g)) \|_\infty \leq \limsup_{n\to\infty} \frac{|F_n \Delta \gamma_0^{-1} F_n|}{|F_n|} = 0.$$

Hence, combining this with the above inequality we have

$$\limsup_{n\to\infty} \varphi(|\frac{1}{|F_n|} \Sigma_{\gamma\in F_n^{-1}} \sigma_\gamma(f)|^2) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this shows that

$$\limsup_{n\to\infty} \| \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \sigma_\gamma(f) \|^2_{L^2(A,\varphi)} = 0.$$

□

Exercise 3.2.3. Generalize the above lemma to arbitrary measure preserving actions. That is to say, suppose $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is a measure preserving action of a countable amenable group Γ on a probability space $(X, \mathcal{B}, \mu)$. Let $F_n \subset \Gamma$ be a Følner sequence, and suppose $A \subset L^\infty(X, \mathcal{B}, \mu)$ is a Γ-invariant unital, self-adjoint subalgebra such that if $L^\infty(X, \mathcal{I}, \mu)$ is the algebra of bounded Γ-invariant functions then $L^\infty(X, \mathcal{I}, \mu) \cap A$ is dense in $L^2(X, \mathcal{I}, \mu)$. Suppose φ is a state on A such that for all $f \in A$ we have

$$\lim_{n\to\infty} \frac{1}{|F_n|} \Sigma_{\gamma\in F_n^{-1}} \varphi(\sigma_\gamma(f)) = \int f \, d\mu.$$

Then show that for all $f \in A$ we have

$$\lim_{n\to\infty} \frac{1}{|F_n|} \Sigma_{\gamma\in F_n^{-1}} \sigma_\gamma(f) = E_{\mathcal{I}}(f),$$

where the convergence is in $L^2(A, \varphi)$.

Theorem 3.2.4. *Let Γ be a countable amenable group with Følner sequence $F_n \subset \Gamma$. Suppose $k \in \mathbb{N}$ and $\Gamma \curvearrowright^{\alpha^j} (X, \mathcal{B}, \mu)$ are weak mixing, measure preserving actions of Γ on a probability space $(X, \mathcal{B}, \mu)$, for $0 \leq j < k$. Assume moreover that the actions pairwise commute, i.e., $\alpha^i_\gamma \circ \alpha^j_\lambda = \alpha^j_\lambda \circ \alpha^i_\gamma$, for all $i \neq j$, and $\gamma, \lambda \in \Gamma$, and that the actions $\gamma \mapsto \alpha^i_\gamma \alpha^{i+1}_\gamma \cdots \alpha^j_\gamma$ are weak mixing for all $0 \leq i \leq j < k$. Suppose $f_0, \ldots, f_{k-1} \in L^\infty(X, \mathcal{B}, \mu)$. Then we have*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \Sigma_{\gamma\in F_n^{-1}} \sigma^0_\gamma(f_0)(\sigma^0_\gamma \sigma^1_\gamma(f_1)) \cdots (\sigma^0_\gamma \cdots \sigma^{k-1}_\gamma(f_{k-1})) = \Pi_{j=0}^{k-1} (\int f_j \, d\mu),$$

where the convergence is in $L^2(X, \mathcal{B}, \mu)$.

Proof. Consider the action $\Gamma \curvearrowright (X^k, \mathcal{B}^{\otimes k}, \mu^k)$ given by $\gamma(x_0, x_1, \ldots, x_{k-1}) = (\alpha^0_\gamma x_0, \alpha^0_\gamma \alpha^1_\gamma x_1, \ldots, \alpha^0_\gamma \cdots \alpha^{k-1}_\gamma x_{k-1})$.

Denote by ν the diagonal measure on X^k given by $\nu(A) = \mu(\{x \in X \mid (x, x, \ldots, x) \in A\})$. Then we obtain a well defined state φ on $A = L^\infty(X, \mathcal{B}, \mu)^{\otimes_{\rm alg} k}$ by

$$\varphi(\tilde{f}) = \int \tilde{f}\, d\nu,$$

for all $\tilde{f} \in A$.

By Lemma 3.2.2, in order to prove the theorem it is enough to show that φ is generic with respect to F_n, and μ^k. We will do by induction on k.

Note that $k = 1$ is trivial. If this holds for $k \geq 1$, and $f_0, f_1, \ldots, f_k \in L^\infty(X, \mathcal{B}, \mu)$, then by Lemma 3.2.2 we have that

$$\lim_{n \to \infty} \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \sigma_\gamma^1(f_1) \cdots (\sigma_\gamma^1 \cdots \sigma_\gamma^k(f_k)) = \Pi_{j=1}^k (\int f_j \, d\mu),$$

in $L^2(X, \mathcal{B}, \mu)$.

Multiplying this by f_0 and integrating we obtain

$$\lim_{n \to \infty} \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \int f_0 \sigma_\gamma^1(f_1) \cdots (\sigma_\gamma^1 \cdots \sigma_\gamma^k(f_k)) \, d\mu = \Pi_{j=0}^k (\int f_j \, d\mu).$$

Applying σ_γ^0 then gives the result. □

Corollary 3.2.5. *Let* $\mathbb{Z} \curvearrowright^T (X, \mathcal{B}, \mu)$ *be a weak mixing, measure preserving action on a probability space* $(X, \mathcal{B}, \mu)$*. Then for each* $k \in \mathbb{N}$*, and* $f_0, f_1, \ldots, f_{k-1} \in L^\infty(X, \mathcal{B}, \mu)$ *we have*

$$\lim_{N \to \infty} \frac{1}{N} \Sigma_{n=0}^{N-1} \sigma_n(f_0) \sigma_{2n}(f_1) \cdots \sigma_{kn}(f_{k-1}) = \Pi_{j=0}^{k-1} (\int f_j \, d\mu),$$

where the convergence is in $L^2(X, \mathcal{B}, \mu)$*.*

Proof. If we consider the action $\alpha^i : \mathbb{Z} \to \mathrm{Aut}(X, \mathcal{B}, \mu)$ which takes the generator of $\mathbb{Z}$ to T^i, then it follows from Corollary 1.7.7 that these actions satisfy the hypotheses of the above theorem. □

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